

# THE CORRECT CLASSIC GENERALIZED LEAST-SQUARES ESTIMATOR OF AN UNKNOWN CONSTANT MEAN OF RANDOM FIELD

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**ABSTRACT.** The aim of the paper is to derive for the negative correlation function with a time parameter an asymptotic disjunction  $\lim_{j \rightarrow \infty} \omega_j^i v_i$  of the numerical generalized least-squares estimator  $\omega_j^i v_i$  of an unknown constant mean of random field in fact the correct classic generalized least-squares estimator of an unknown constant mean of the field.

## 1. INTRODUCTION

The best linear unbiased generalized (estimation) statistics  $\hat{V}_j = \sum_{i=1}^n \omega_j^i V_i = \omega_j^i V_i$  of the random field  $V_j$ ;  $j \subset i = 1, \dots, n$  at  $j \geq n+1$  with unknown constant mean  $m$  and variance  $\sigma^2$  that fulfils the constraint

$$\lim_{j \rightarrow \infty} E\{[V_j - \omega_j^i V_i]^2\} = \sigma^2 = E\{[V_j - m]^2\}$$

is the classic best linear unbiased generalized statistics for finite  $n$  and  $j \rightarrow \infty$  of an unknown constant mean  $m = E\{V_j\}$  of the field  $V_j$  with the classic generalized least-squares estimator  $\lim_{j \rightarrow \infty} \omega_j^i v_i$  of an unknown constant mean of the field and with constrained variance of the best linear unbiased generalized (estimation) statistics  $\lim_{j \rightarrow \infty} E\{[\omega_j^i V_i - m]^2\}$  of the field as its variance (a mean squared error of mean estimation).

The best linear unbiased generalized (estimation) statistics that fulfils (on computer) the constraint

$$E\{[V_j - \omega_j^i V_i]^2\} = \sigma^2 = E\{[V_j - m]^2\}$$

is the numerical best linear unbiased generalized statistics for finite  $n$  at finite  $j$  of an unknown constant mean  $m = E\{V_j\}$  of the field  $V_j$  with the numerical generalized least-squares estimator  $\omega_j^i v_i$  of an unknown constant mean of the field and with constrained variance of the best linear unbiased generalized (estimation) statistics  $E\{[\omega_j^i V_i - m]^2\}$  of the field as its variance.

Since the classic best linear unbiased generalized statistics for finite  $n$  and  $j \rightarrow \infty$  of an unknown constant mean  $m = E\{V_j\}$  of the field  $V_j$  is an asymptotic disjunction for  $j \rightarrow \infty$  of the numerical best linear unbiased generalized statistics for finite  $n$  at finite  $j$  of an unknown constant mean  $m = E\{V_j\}$  of the field  $V_j$  then the correct classic generalized least-squares estimator  $\lim_{j \rightarrow \infty} \omega_j^i v_i$  of an unknown constant mean  $m$  of the field is an asymptotic disjunction for  $j \rightarrow \infty$  of the numerical generalized least-squares estimator  $\omega_j^i v_i$  of an unknown constant mean  $m$  of the field.

## 2. THE CORRELATION FUNCTION AND THE ESTIMATORS

From the so-called semi-variogram  $\gamma(h)$  for a stationary random field  $V_j$  with an unknown constant mean  $m = E\{V_j\}$ , variance  $\sigma^2 = E\{V_j^2\} - E^2\{V_j\}$  and covariance function  $C(h) = E\{V_j V_{j+h}\} - E\{V_j\}E\{V_{j+h}\}$

$$\begin{aligned}\gamma(h) &= \frac{1}{2}E\{(V_j - V_{j+h})^2\} \\ &= \frac{1}{2}E\{V_j^2 - 2V_j V_{j+h} + V_{j+h}^2\} \\ &= E\{V_j^2\} - E\{V_j V_{j+h}\} \\ &= E\{V_j^2\} - E^2\{V_j\} - (E\{V_j V_{j+h}\} - E^2\{V_j\}) \\ &= E\{V_j^2\} - E^2\{V_j\} - (E\{V_j V_{j+h}\} - E\{V_j\}E\{V_{j+h}\}) \\ &= \sigma^2 - C(h) \geq 0\end{aligned}$$

we get the absolute value of the correlation function  $\rho(h)$

$$|\rho(h)| = C(h)/C(0) = C(h)/\sigma^2 = 1 - \gamma(h)/\sigma^2.$$

Since the correlation function is non-increasing then the (first) experimental correlogram

$$(1) \quad |\hat{\rho}(h)| = 1 - \hat{\gamma}(h)/\hat{\sigma}^2$$

should be computed for non-decreasing outcomes for  $h \leq d$

$$\hat{\sigma}^2 = \hat{\gamma}(d) \geq \hat{\gamma}(d-1) \geq \hat{\gamma}(d-2) \geq \dots \geq \hat{\gamma}(1) > \hat{\gamma}(0) = 0$$

of the experimental semi-variogram for a time series of the length  $n$

$$\hat{\gamma}(h) = \frac{1}{2} \frac{1}{(n-h)} \sum_{j=1}^{n-h} (v_j - v_{j+h})^2 \quad h = 0, \dots, n-1.$$

On the other hand from definition of the covariance function

$$C(h) = E\{V_j V_{j+h}\} - E\{V_j\}E\{V_{j+h}\}$$

we get for a time series of the length  $n$

$$\hat{C}(h) = \frac{1}{n-h} \sum_{j=1}^{n-h} v_j v_{j+h} - \frac{1}{(n-h)^2} \sum_{j=1}^{n-h} v_j \sum_{j=1}^{n-h} v_{j+h}$$

and the (second) experimental correlogram

$$(2) \quad |\hat{\rho}(h)| = \hat{C}(h)/\hat{C}(0)$$

for  $h = 0, \dots, d$ .

## 3. THE CORRECT CLASSIC GENERALIZED LEAST-SQUARES ESTIMATOR OF AN UNKNOWN CONSTANT MEAN OF THE FIELD

The attached source code “combo.pas” let us find (see Tab. 1) for the negative correlation function with the time parameter  $t = n+1, \dots, n+s$

$$(3) \quad \rho(\Delta_{ij}) = \begin{cases} -1 \cdot t^{-\Theta[\Delta_{ij}/t]^2}, & \text{for } h = \Delta_{ij} = |i-j| > 0, \\ +1, & \text{for } h = \Delta_{ij} = |i-j| = 0, \end{cases}$$

where  $\Theta$  is derived by Levenberg-Marquardt fit

$$|\rho(h)| = n^{-\Theta[h/n]^2},$$

to the experimental correlograms (1) and (2), the asymptotic disjunction  $\lim_{j \rightarrow \infty} \omega_j^i v_i$  of the numerical generalized least-squares estimator  $\omega_j^i v_i$  of an unknown constant mean  $m$  of the field  $V_j$ ;  $j \subset i = 1, \dots, n$  in fact the correct classic generalized least-squares estimator of an unknown constant mean  $m$  of the field.

The variance of the best linear unbiased generalized (estimation) statistics

$$(4) \quad E\{[\omega_j^i V_i - m]^2\} = -\sigma^2(\omega_j^i \rho_{ij} - \mu_j^1)$$

under the constraint

$$E\{[V_j - \omega_j^i V_i]^2\} = \sigma^2 = E\{[V_j - m]^2\}$$

given by the numerical approximation to the root of equation

$$(5) \quad \omega_j^i \rho_{ij} + \mu_j^1 = 0,$$

where (kriging algorithm)

$$\underbrace{\begin{bmatrix} \omega_j^1 \\ \vdots \\ \omega_j^n \\ \mu_j^1 \end{bmatrix}}_{(n+1) \times 1} = \underbrace{\begin{bmatrix} \rho_{11} & \dots & \rho_{1n} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ \rho_{n1} & \dots & \rho_{nn} & 1 \\ 1 & \dots & 1 & 0 \end{bmatrix}}_{(n+1) \times (n+1)}^{-1} \cdot \underbrace{\begin{bmatrix} \rho_{1j} \\ \vdots \\ \rho_{nj} \\ 1 \end{bmatrix}}_{(n+1) \times 1},$$

becomes a variance of the best linear unbiased generalized statistics of an unknown constant mean  $m$  of random field with 95% confidence intervals for a mean estimation (from (4))

$$\hat{m} = \omega_j^i v_i \pm 1.96 \sqrt{-\hat{\sigma}^2(\omega_j^i \rho_{ij} - \mu_j^1)},$$

where

$$\hat{\sigma}^2 = \omega_j^i v_i^2 - (\omega_j^i v_i)^2,$$

if (5) holds.

Fig.	Index	$n$	$\Theta$	$t = n + s$	$j = u$	$\omega_j^i v_i$
1	FTSE 100	132	0.83283	238	426	8463.42
2	DJI	113	0.93670	203	356	13603.87
3	S&P 500	102	0.93569	174	270	1788.80

TABLE 1. The numerical generalized least-squares estimator  $\omega_j^i v_i$  of an unknown constant mean  $m = E\{V_j\}$  of the field  $V_j$  for the negative correlation function (3) with the time parameter  $t = n + 1, \dots, n + s$  as the correct classic generalized least-squares estimator of an unknown constant mean of the field for final  $t = n + s$  at final  $j = u$ .

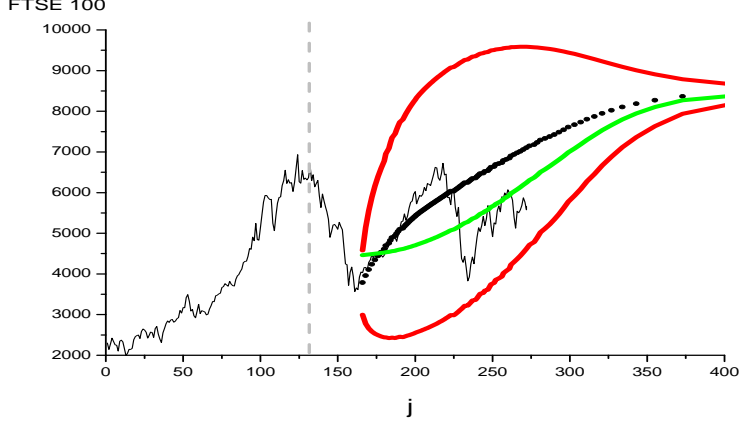


FIGURE 1. FTSE 100 from 1 September 1989 up to 1 May 2012 (272 monthly close quotes). The numerical (black dots) generalized least-squares estimator  $\omega_j^i v_i$  of an unknown constant mean of the field with 95% confidence intervals (red lines) for a mean estimation is compared for the negative correlation function (3) with the parameter  $t = n + 1, \dots, n + s$  at finite  $j \geq n + 1$  to the classic (grey line) generalized least-squares estimator of an unknown constant mean  $\lim_{j \rightarrow \infty} \omega_j^i v_i$  of the field. The asymptotic limit of the classic generalized least-squares statistics of an unknown constant mean of the field is fulfilled for final  $t = 238$  at final  $j = 426$ . Dashed vertical line denotes  $n = 132$ .

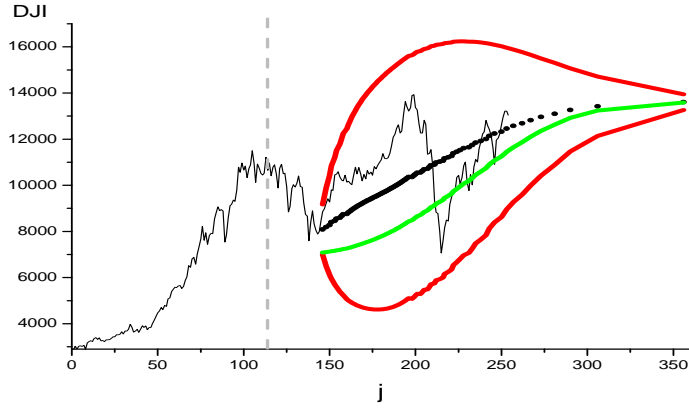


FIGURE 2. DJI from 1 April 1991 up to 1 May 2012 (254 monthly close quotes). The same as in Fig. 1. The asymptotic limit of the classic generalized least-squares statistics of an unknown constant mean of the field is fulfilled for final  $t = 203$  at final  $j = 356$ . Dashed vertical line denotes  $n = 113$ .

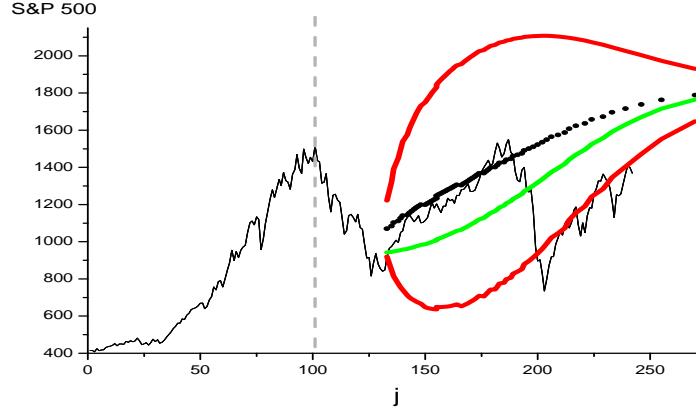


FIGURE 3. S&P 500 from 3 April 1992 up to 1 May 2012 (242 monthly close quotes). The same as in Fig. 1. The asymptotic limit of the classic generalized least-squares statistics of an unknown constant mean of the field is fulfilled for final  $t = 174$  at final  $j = 270$ . Dashed vertical line denotes  $n = 102$ .

#### REFERENCES

- [1] E. H. Isaaks and R. M. Srivastava, *An Introduction to Applied Geostatistics*, New York: Oxford Univ. Press (1989).
  - [2] T. Susło, *The Numerical Generalized Least-Squares Estimator of an Unknown Constant Mean of Random Field*, [arXiv:1111.3971 \[cs.NA\]](#).
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